THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2020/21 Term 2

Homework 3

Updated due date: 5th March 2021

All questions are selected from the textbook. Please **submit online through Blackboard** your answers to Compulsory Part only. The late submission will not be accepted. Reference solutions to both parts will be provided after grading.

Compulsory Part
Chapter 1 (page 41): 15, 18, 20(b), 24, 26, 27, 29, 32, 34, 36(a)

Optional Part
Chapter 1 (Page 41): 17, 23, 25, 28, 30, 31, 33, 35, 36(b)(c)(d), 37, 38

Compulsory Part:

15. Proof. It is clear when $x = y$. If $x \neq y$,

$$\sum_{n=0}^{\infty} P^n(x,y) = \sum_{n=1}^{\infty} P^n(x,y) = G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}$$

$$\leq \frac{1}{1-\rho_{yy}} = 1 + \frac{\rho_{yy}}{1-\rho_{yy}} = 1 + G(y,y) = \sum_{n=0}^{\infty} P^n(y,y).$$

18.(a) Proof. For two nonnegative integers x and y, we have

$$P^{y+1}(x,y) > P(x,0)P(0,1)P(1,2)\cdots P(y-1,y) = (1-p)p^y > 0.$$

By Q16, x leads to y. Hence the chain is irreducible.

(b) Solution. For $n = 1$, $P_0(T_0 = 1) = P(0,0) = 1-p$.
For $n \geq 2$, $P_0(T_0 = n) = P(0,1)P(1,2)\cdots P(n-2,n-1)P(n-1,0) = p^{n-1}(1-p)$.

(c) Proof. Note that $\rho_{00} = \sum_{n=1}^{\infty} P_0(T_0 = n) = \sum_{n=1}^{\infty} p^{n-1}(1-p) = 1$. This implies that 0 is recurrent. Since the chain is irreducible, it is recurrent.

20. Solution. (a) There are two irreducible closed sets $C_1 = \{0,1\}$ and $C_2 = \{2,4\}$. Hence $3, 5$ are transient and $0, 1, 2, 4$ are recurrent.

(b) Clearly $\rho_{\{0,1\}}(0) = \rho_{\{0,1\}}(1) = 1$ and $\rho_{\{0,1\}}(2) = \rho_{\{0,1\}}(4) = 0$. By one-step argument, we have

$$\begin{cases} \rho_{\{0,1\}}(3) = 1/2 + (1/4)\rho_{\{0,1\}}(5), \\ \rho_{\{0,1\}}(5) = 1/5 + (1/5)\rho_{\{0,1\}}(3) + (2/5)\rho_{\{0,1\}}(5). \end{cases}$$

Hence $\rho_{\{0,1\}}(3) = 7/11$ and $\rho_{\{0,1\}}(5) = 6/11$.

24. Solution. Let X_n denote the capital of the gambler at time n, with $X_0 = x$, where $0 < x < d$. The transition function is

$$P(x,y) = \begin{cases} p, & y = x+1; \\ q = 1-p, & y = x-1; \\ 0, & \text{otherwise}, \end{cases}$$

for $0 < x < d$.

Since the gambler's game is a special case of birth and death chains, we can use (59) (on textbook, page 31) or calculate directly by solving difference equations:

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y},$$

where $\gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y}$ and $a < x < b$. In this gambler ruin problem

$$\gamma_y = \left(\frac{q}{p}\right)^y.$$

Put $a = 0$ and $b = d$, and $0 < x < d$,

$$P_x(T_0 < T_d) = \frac{\sum_{y=x}^{d-1} (\frac{q}{p})^y}{\sum_{y=0}^{d-1} (\frac{q}{p})^y} = \begin{cases} \dfrac{(\frac{q}{p})^x - (\frac{q}{p})^d}{1 - (\frac{q}{p})^d}, & p \neq \frac{1}{2}; \\ \dfrac{d - x}{d}, & p = \frac{1}{2}. \end{cases}$$

26. Proof. Using (59) (on textbook, page 31), we have

$$P_x(T_0 < T_n) = \frac{\sum_{y=x}^{n-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y},$$

for $0 < x < n$. Note that for $x > 0$, $1 \le T_{x+1} < T_{x+2} < \cdots$. Hence $\{T_0 < T_n\}_{n=1}^{\infty}$ forms a nondecreasing sequence of events. By continuity of the probability, we have for $x \ge 1$,

$$\rho_{x0} = P_x(T_0 < \infty) = P_x\left(\bigcup_{n=1}^{\infty} \{T_0 < T_n\}\right) = \lim_{n\to\infty} P_x(T_0 < T_n) = 1 - \lim_{n\to\infty} \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y}.$$

(a) If $\sum_{y=0}^{\infty} \gamma_y = \infty$, then the above limit is 0 and $\rho_{x0} = 1$.

(b) If $\sum_{y=0}^{\infty} \gamma_y < \infty$, then

$$\rho_{x0} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y}.$$

27. Proof. (a) If $q \ge p$, then

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \left(\frac{q}{p}\right)^y \ge \sum_{y=0}^{\infty} 1^y = \infty.$$

Hence by Q26(a), $\rho_{x0} = 1$.

(b) If $q < p$, then

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \left(\frac{q}{p}\right)^y = \frac{1}{1 - \frac{q}{p}} = \frac{p}{p - q} < \infty.$$

Hence by Q26(b) and $\sum_{y=x}^{\infty} \gamma_y = (q/p)^x \cdot p/(p - q)$,

$$\rho_{x0} = \frac{(q/p)^x \cdot p/(p - q)}{p/(p - q)} = (q/p)^x.$$

29. (a) Proof. Note that for $y \ge 1$,

$$\gamma_y = \prod_{x=1}^{y} \frac{q_x}{p_x} = \frac{1^2 \cdot 2^2 \cdots y^2}{2^2 \cdots y^2 \cdot (y + 1)^2} = \frac{1}{(y + 1)^2}.$$

Therefore, $\sum_{y=0}^{\infty} \gamma_y = 1 + \sum_{y=1}^{\infty} \frac{1}{(y+1)^2} = \frac{\pi^2}{6} < \infty$. Hence the chain is transient.

(b) Solution. By Q26(b),

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y} = 1 - \frac{6}{\pi^2} \sum_{y=0}^{x-1} \frac{1}{(y + 1)^2}.$$

32. Solution. Note that in Example 14, the probability that the male line of a given man eventually becomes extinct is $\rho = \sqrt{5} - 2$. Hence if $X_1 = 2$, the probability that the male line will continue forever is

$$1 - \rho^2 = 4(\sqrt{5} - 2) \approx 0.9443.$$

34. Proof. The mean number of offspring is

$$\mu = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} px(1-p)^x = \frac{1-p}{p}.$$

If $p \geq 1/2$, then $\mu \leq 1$ and so $\rho = 1$.

If $p < 1/2$, then $\mu > 1$. We need to solve

$$t = \sum_{y=0}^{\infty} p(1-p)^y t^y = \frac{p}{1-(1-p)t},$$

or equivalently,

$$(1-p)t^2 - t + p = 0.$$

This equation has two roots 1 and $\frac{p}{1-p}$. Consequently, $\rho = \frac{p}{1-p}$.

36. Proof. **(a)**

$$\begin{aligned}
E[X_{n+1}^2 \mid X_n = x] &= E[(\xi_1 + \xi_2 + \cdots + \xi_x)^2] \\
&= \sum_{i=1}^{x} E(\xi_i^2) + 2 \sum_{1 \leq i < j \leq x} E(\xi_i \xi_j) \\
&= \sum_{i=1}^{x} (E(\xi_i^2) - (E\xi_i)^2) + (\sum_{i=1}^{x} E\xi_i)^2 \\
&= x\sigma^2 + x^2\mu^2.
\end{aligned}$$

Optional Part

17. Proof. By Q16, there exists $n, m \in \mathbb{Z}_+$ such that $P^n(x,y) > 0$ and $P^m(y,z) > 0$. Then $P^{n+m}(x,z) \geq P^n(x,y)P^m(y,z) > 0$. Hence by Q16, x leads to z.

23. Solution. Since $\binom{2d}{y} = \frac{2d}{2d-y}\binom{2d-1}{y}$, we have

$$\begin{aligned}\sum_{y=0}^{2d} P(x,y)\frac{2d-y}{2d} &= \sum_{y=0}^{2d} \binom{2d-1}{y}\left(\frac{x}{2d}\right)^y\left(1-\frac{x}{2d}\right)^{2d-y} \\ &= \frac{2d-x}{2d}\sum_{y=0}^{2d-1}\binom{2d-1}{y}\left(\frac{x}{2d}\right)^y\left(1-\frac{x}{2d}\right)^{2d-1-y} \\ &= \frac{2d-x}{2d}.\end{aligned}$$

Compare with the one-step formula

$$\rho_{\{0\}}(x) = \sum_{y=0}^{2d} P(x,y)\rho_{\{0\}}(y).$$

Hence $\rho_{\{0\}}(x) = \frac{2d-x}{2d}$, $0 < x < 2d$.

25. Solution. **(a)** In Q24, let $p = 9/19$, $q = 10/19$, $d = 1001$ and $x = 1000$. Then

$$P_{1000}(T_0 < T_{1001}) = \frac{(\frac{10}{9})^{1001} - (\frac{10}{9})^{1000}}{(\frac{10}{9})^{1001} - 1} \approx 0.1.$$

(b) The expected loss is

$$1000 \cdot P_{1000}(T_0 < T_{1001}) - P_{1000}(T_0 > T_{1001}) \approx 100 - 0.9 = 99.1.$$

28. Proof. If $p_x \leq q_x$, $x \geq 1$, then

$$\sum_{y=0}^{\infty}\gamma_y = 1 + \sum_{y=1}^{\infty}\frac{q_1\cdots q_y}{p_1\cdots p_y} \geq 1 + \sum_{y=1}^{\infty}1^y = \infty.$$

Hence by Q26(a), $\rho_{10} = 1$. By one-step argument, we have

$$\rho_{00} = P(0,0)\rho_{00} + P(0,1)\rho_{10} = r_0\rho_{00} + p_0.$$

Since $p_0 + r_0 = 1$ and $p_0 > 0$, we have $\rho_{00} = 1$, that is, state 0 is recurrent. As the chain is irreducible, it is recurrent.

30. Solution. **(a)** Note that in Example 13, $\gamma_x = 2(\frac{1}{x+1} - \frac{1}{x+2})$. By (59), for $a < x < b$,

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1}\gamma_y}{\sum_{y=a}^{b-1}\gamma_y} = \frac{2(\frac{1}{x+1} - \frac{1}{b+1})}{2(\frac{1}{a+1} - \frac{1}{b+1})} = \frac{(a+1)(b-x)}{(x+1)(b-a)}.$$

(b) By Q26(b), for $x > 0$,

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty}\gamma_y}{\sum_{y=0}^{\infty}\gamma_y} = \frac{\frac{2}{x+1}}{2} = \frac{1}{x+1}.$$

31. Proof. If $f(0) > 0$, then for any $x > 0$,

$$P(x, 0) = f(0)^x > 0.$$

Since 0 is absorbing, any positive x is transient.

If $f(0) = 0$, then X_n is nondecreasing, that is, $\rho_{xy} = 0$ for $x > y$. Moreover, for $x > 0$,

$$\rho_{xx} = P(x, x) = f(1)^x < 1.$$

Hence any positive x is transient.

33. Solution. The mean number of offspring of one given particle is $\mu = 3/2 > 1$. Hence the extinction probability ρ is the root of the equation

$$\frac{1}{2} + \frac{1}{2}t^3 = t$$

lying in $[0, 1)$. We can rewrite this equation as

$$(t - 1)(t^2 + t - 1) = 0.$$

This equation has three roots, namely, 1, $\frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$. Consequently, $\rho = \frac{-1+\sqrt{5}}{2}$.

35. Proof. Note that for $x \geq 1$,

$$\sum_y yP(x, y) = E_x(X_1) = E(\xi_1 + \xi_2 + \cdots + \xi_x) = xE(\xi_1) = \mu x.$$

Using Q13(b), we have $E_x(X_n) = \mu^n E_x(X_0) = x\mu^n$.

36.

(b) Using Total Expectation Formula, Q36(a) and Q35, we have

$$\begin{aligned}
E_x(X_{n+1}^2) &= \sum_y P_x(X_n = y)E[X_{n+1}^2 \mid X_n = y] \\
&= \sum_y P_x(X_n = y)(y\sigma^2 + y^2\mu^2) \\
&= \sigma^2 \sum_y yP_x(X_n = y) + \mu^2 \sum_y y^2 P_x(X_n = y) \\
&= \sigma^2 E_x(X_n) + \mu^2 E_x(X_n^2) \\
&= x\mu^n\sigma^2 + \mu^2 E_x(X_n^2).
\end{aligned}$$

(c) Use induction on n. For $n = 1$, using Q36(a), we have

$$E_x(X_1^2) = x\sigma^2 + x^2\mu^2.$$

Suppose that the formula holds for some $n \geq 1$, then

$$\begin{aligned}
E_x(X_{n+1}^2) &= x\mu^n\sigma^2 + \mu^2 E_x(X_n^2) \\
&= x\mu^n\sigma^2 + \mu^2(x\sigma^2(\mu^{n-1} + \cdots + \mu^{2(n-1)}) + x^2\mu^{2n}) \\
&= x\sigma^2(\mu^n + \cdots + \mu^{2n}) + x^2\mu^{2(n+1)}.
\end{aligned}$$

Hence the formula also holds for $n + 1$.

(d) If there are x particles initially, then by Q35 and Q36(c), for $n \geq 1$,

$$\mathrm{Var} X_n = E_x(X_n^2) - (E_x(X_n))^2 = \begin{cases} x\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right), & \mu \neq 1, \\ nx\sigma^2, & \mu = 1. \end{cases}$$

37. Proof. (a) If $f(0) = 0$, then $P(x, x-1) = f(0) = 0$ for $x \geq 1$. That implies $\rho_{xy} = 0$ for $x > y \geq 0$. Hence the chain is not irreducible.

If $f(0) + f(1) = 1$, then $P(x, y) = f(y - x + 1) = 0$ for $1 \leq x < y$. That implies $\rho_{xy} = 0$ for $1 \leq x < y$. Hence the chain is not irreducible.

(b) For $x > y \geq 0$,

$$\rho_{xy} \geq P(x, x-1)P(x-1, x-2)\cdots P(y+1, y) = (f(0))^{x-y} > 0.$$

Since $f(0) + f(1) < 1$, there exists $x_0 \geq 2$ such that $f(x_0) > 0$. Then for $n \geq 0$,

$$\begin{aligned} \rho_{0,x_0+n(x_0-1)} &\geq P(0, x_0)P(x_0, x_0 + (x_0 - 1))P(x_0 + (x_0 - 1), x_0 + 2(x_0 - 1))\cdots \\ &\quad P(x_0 + (n-1)(x_0 - 1), x_0 + n(x_0 - 1)) \\ &= f(x_0)^{n+1} > 0. \end{aligned}$$

Now for any states x, y, there exists n such that $x_0 + n(x_0 - 1) > y$. Since x leads to 0, 0 leads to $x_0 + n(x_0 - 1)$, $x_0 + n(x_0 - 1)$ leads to y, x also leads to y. Hence the chain is irreducible.

38. Solution. (a) If $f(1) = 1$, all positive states $1, 2, \ldots$ are absorbing and recurrent, while 0 is transient.

(b) If $f(0) > 0$, $f(1) > 0$, and $f(0) + f(1) = 1$, states 0 and 1 are recurrent, while $2, 3, \ldots$ are transient.

(c) If $f(0) = 1$, state 0 is absorbing and recurrent, while $1, 2, \ldots$ are transient.

(d) If $f(0) = 0$ and $f(1) < 1$, all states are transient.